

The role of zero modes for the infrared behavior of QCD

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Abstract

We analyse the mechanism in which zero modes lead to an elimination of fermionic color non-singlet states in 1+1 dimensions. Using a hamiltonian lattice formulation we clarify the physical meaning of the zero modes but we do not find support for speculations on the crucial importance of lower dimensional fields (zero modes in 1+1 dimension) for the infrared behavior of QCD in 2+1 or higher dimensions.

1 Introduction

Within the last years, formulations of the QCD hamiltonian in terms of gauge invariant degrees of freedom have been given [1, 2]. In such approaches the hamiltonian contains singular coupling terms which resemble in many cases centrifugal barriers, depending, however, on fields rather than on quantum mechanical variables. In particular in an axial gauge representation of QCD these singular coupling terms were found to be associated with "lower dimensional fields" which are zero modes with respect to one coordinate direction [2]. Due to this particular interaction a calculation in 1+1 dimensional Yang-Mills theory in the presence of static charges has shown that the infrared properties of the model crucially depend on these zero modes [3]. Similar observations have also been made in a study of Yang-Mills theory coupled to non-relativistically moving particles [4]. In view of these findings, the question has been raised as to whether these lower dimensional fields may play a crucial role for understanding the infrared properties of QCD and in particular confinement in 3+1 dimensions, as well [2, 3].

In this letter we intend to address this question in the framework of the lattice hamiltonian formulation of $SU(N)$ Yang-Mills theory in the presence of static color sources in 1+1 dimensions. In contrast to the continuum formulation used in the aforementioned approaches we have a simple physical interpretation for the gauge invariant degrees of freedom in the lattice formulation. This enables us to interpret the "lower dimensional fields" which become quantum mechanical variables in 1+1 dimension. Therefore an understanding of the role of zero modes can be gained which is general enough to allow us drawing conclusions about higher dimensions.

2 Hamiltonian lattice QCD in 1+1 dimensions

The variables in the hamiltonian formalism on the lattice are the link variables $\hat{U}(l)$; $l = 1 \dots M$, which are group elements of $SU(N)$, and corresponding "angular momentum" operators $\hat{J}_R^a(l)$

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and $\hat{J}_L^a(l)$ which generate right and left multiplication in the corresponding representation of the SU(N) group. They satisfy the commutation relations

$$\begin{aligned} [\hat{J}_L^a(l), \hat{U}_{ij}(l')] &= (-T^a \hat{U})_{ij}(l) \delta_{l,l'} & [\hat{J}_L^a(l), \hat{J}_L^b(l')] &= i f^{abc} \hat{J}_L^c(l) \delta_{l,l'} \\ [\hat{J}_R^a(l), \hat{U}_{ij}(l')] &= (-\hat{U} T^a)_{ij}(l) \delta_{l,l'} & [\hat{J}_R^a(l), \hat{J}_R^b(l')] &= -i f^{abc} \hat{J}_R^c(l) \delta_{l,l'} \\ [\hat{J}_L^a(l), \hat{J}_L^b(l')] &= 0 \end{aligned} \quad (1)$$

where f^{abc} is the structure constant of the group and we choose the normalization $2Tr\{T^a T^b\} = \delta^{a,b}$. $\hat{J}_R^a(l)$ and $\hat{J}_L^a(l)$ are related using the adjoint representation $D_{ab}^{(1)}$

$$\hat{J}_L^a(l) = D_{ba}^{(1)}(U(l)) \hat{J}_R^b(l) = 2Tr[U(l) T^b U^\dagger(l) T^a] \hat{J}_R^b(l) \quad (2)$$

The standard Kogut-Susskind lattice hamiltonian [5], which we shall use, reduces in two dimensions and in the absence of dynamical fermions to the sum of the quadratic casimir operators

$$\hat{H} = \frac{g_L^2}{2a} \sum_{link} \hat{J}_R^a(l) \hat{J}_R^a(l) = \frac{g_L^2}{2a} \sum_{link} \hat{J}_L^a(l) \hat{J}_L^a(l), \quad (3)$$

where g_L is the dimensionless lattice coupling constant and a is the lattice spacing. The hamiltonian is invariant under time independent gauge transformations generated by Gauss law operators $\hat{G}^a(l)$

$$\hat{G}^a(l) = \{\hat{J}_L^a(l) - \hat{J}_R^a(l+1) + \psi^\dagger(l) T^a \psi(l)\}, \quad (4)$$

through which static color sources enter the formulation. Since physical states have to be gauge invariant, we have to impose the constraints

$$\hat{G}^a(l) |phys\rangle = 0. \quad (5)$$

In the following we consider only the case with one static quark and one static anti-quark, which will also provide the solution in the case without static sources in the limit of zero spatial separation.

3 The spectrum of states

We choose the sites 0 and m to be occupied with the heavy quark and the heavy anti-quark respectively. In order to find the spectrum of gauge invariant states we introduce the products A and B of matrices $U(l)$

$$\begin{aligned} A &= U(m) \times \cdots \times U(1) \\ B &= U(M) \times \cdots \times U(m+1). \end{aligned} \quad (6)$$

These particular combinations are useful since it is obvious that by combining A , B and static sources gauge invariant states can be obtained. For example when acting on the gauge invariant ground state $|0\rangle$ of eq.(3) with the property

$$\hat{J}_R^a(l) |0\rangle = 0 \quad \left(\hat{J}_L^a(l) |0\rangle = 0 \right) \quad (7)$$

gauge invariant states are created from the operators $\psi^\dagger \hat{A} \psi$, $\psi^\dagger \hat{B}^\dagger \psi$, $\psi^\dagger \hat{A} \hat{B} \hat{A} \psi$, $Tr[\hat{A} \hat{B}] \psi^\dagger \hat{A} \psi$ etc.. In order to be able to treat $(U(1), \dots, U(m-1), U(m+1), \dots, U(M-1), A, B)$ as

independent variables, the angular momentum operators must be transformed accordingly. Decomposing A, B in terms of particular link variables $U(l)$ and auxiliary matrices $L(l), R(l)$

$$A = L_A(l)U(l)R_A(l) , \quad B = L_B(l)U(l)R_B(l) , \quad (8)$$

the operators $J_R^a(l)$ are modified in the following way

$$\begin{aligned} l \in [1, m-1] & \quad J_R^a(l) \mapsto J_R^a(l) + D_{ba}^{(1)}(R_A(l))J_R^b(A) \\ l = m & \quad J_R^a(l) \mapsto D_{ba}^{(1)}(R_A(l))J_R^b(A) \\ l \in [m+1, M-1] & \quad J_R^a(l) \mapsto J_R^a(l) + D_{ba}^{(1)}(R_B(l))J_R^b(B) \\ l = M & \quad J_R^a(l) \mapsto D_{ba}^{(1)}(R_B(l))J_R^b(B). \end{aligned} \quad (9)$$

In the new set of variables the constraints (4) now read

$$(l \neq 0, m) \quad \left\{ \begin{array}{l} J_L^a(l) | \text{phys} \rangle = 0 \\ J_R^a(l) | \text{phys} \rangle = 0 \end{array} \right. \quad (10)$$

$$\{J_L^a(A) - J_R^a(B) + \psi^\dagger(m)T^a\psi(m)\} | \text{phys} \rangle = 0 \quad (11)$$

$$\{J_L^a(B) - J_R^a(A) + \psi^\dagger(0)T^a\psi(0)\} | \text{phys} \rangle = 0 \quad (12)$$

which tell us that only the variables A and B , as expected, are relevant to construct physical states. Owing to the replacement (9), the hamiltonian takes the form

$$\hat{H} = \frac{g_L^2 d}{2a^2} \hat{J}_R^a(A) \hat{J}_R^a(A) + \frac{g_L^2 (L-d)}{2a^2} \hat{J}_R^a(B) \hat{J}_R^a(B) + Q, \quad (13)$$

where $d = ma$ ($L = Ma$) is the distance between the two sources and the operator Q has the properties

$$Q | \text{phys} \rangle = 0; \quad [Q, J_R^a(A)J_R^a(A)] = [Q, J_R^a(B)J_R^a(B)] = 0. \quad (14)$$

The constraints, which remain to be implemented, are given in eq.(11,12). The problem of finding the spectrum of physical states has therefore been reduced to that of two "link" variables of length d and $L-d$ respectively. Due to the one dimensional geometry the local degrees of freedom have disappeared by imposing gauge invariance.

To find the eigenvalues and the eigenfunctions of the hamiltonian (13) we restrict our consideration for simplicity to $SU(2)$. The hamiltonian is a sum of quadratic casimir operators which mutually commute. Therefore, in $SU(2)$, the eigenfunctions have the form of a direct product (j_A, j_B) , labeled by an A-spin, j_A , and a B-spin, j_B , respectively. Owing to the constraints (11,12), these two representations, however, have to fulfill the requirement $|j_A - j_B| = \frac{1}{2}$. As an expansion in terms of characters¹ $\hat{\chi}_{\frac{k}{2}}(BA)$ satisfying

$$\hat{J}_R^a(U) \hat{J}_R^a(U) \hat{\chi}_j(U) |0\rangle = j(j+1) \hat{\chi}_j(U) |0\rangle \quad (15)$$

we find the following expressions ($n = 0, 1, 2, \dots$) [6]

$$|(\frac{n}{2} + \frac{1}{2}, \frac{n}{2})\rangle_{\text{phys}} = \sum_{k=1}^n \hat{\chi}_{\frac{k}{2}}(\hat{B}\hat{A}) [\hat{A}(\hat{B}\hat{A})^{(n-k)}]_{ij} \psi_i^\dagger(m) \psi(0)_j |0\rangle \quad (16)$$

¹There is another way of formulating the eigenfunctions of the hamiltonian in terms of D-functions which is less intuitive for our purposes but is easier to generalize to any $SU(N)$ group.

$$|(\frac{n}{2}, \frac{n}{2} + \frac{1}{2})\rangle_{phys} = \sum_{k=1}^n \hat{\chi}_{\frac{k}{2}}(\hat{B}\hat{A}) \left[\hat{A}[(\hat{B}\hat{A})^{(n-k+1)}]^\dagger \right]_{ij} \psi^\dagger(m)_i \psi_j(0) |0\rangle \quad . \quad (17)$$

The eigenvalues corresponding to these states are

$$E(\frac{n}{2} + \frac{1}{2}, \frac{n}{2}) = \frac{g_L^2}{8a^2} [d(n+1)(n+3) + (L-d)n(n+2)] \quad (18)$$

$$E(\frac{n}{2}, \frac{n}{2} + \frac{1}{2}) = \frac{g_L^2}{8a^2} [(L-d)(n+1)(n+3) + d \cdot n(n+2)] \quad . \quad (19)$$

Since in this model the continuum limit can be taken trivially ($g_L/a \rightarrow g_{continuum}$) we find for SU(2) identical results as obtained in [7] and which have also been found in the standard continuum formulation [3].

This agreement also holds in the case of pure Yang-Mills theory which is obtained from the above result in the limit $d \rightarrow 0$ ($A_{ij} \rightarrow \delta_{ij}$). We find the following spectrum and eigenstates (the states (17) still couple to the sources in this limit and therefore they do not describe pure gauge theory)

$$| \frac{n}{2} \rangle_{phys} = \hat{\chi}_{\frac{n}{2}}(\hat{B}\hat{A}) | 0 \rangle , \quad E(\frac{n}{2}) = \frac{g_L^2 L}{2a^2} \frac{n}{2} \frac{n+2}{2} \quad (20)$$

given simply by the characters $\hat{\chi}_{\frac{n}{2}}(\hat{B}\hat{A})$ of SU(2) and agreeing with results obtained in [8].

4 Comparison with the continuum formulation

In this section we intend to compare our lattice results with those of the continuum formulation not only for the spectrum but also for the states themselves. This enables us to interpret the zero mode variables in the continuum approach and to draw conclusions about the importance of related "lower dimensional fields" in higher dimensions. We start by observing that the states (16,17) are characterized only by the combinations of $\hat{B}\hat{A}$. We therefore perform a transformation from \hat{A}, \hat{B} to the variables $\hat{A}, \hat{U} = \hat{B}\hat{A}$ in analogy to the transformation we carried out in the previous section. The hamiltonian and the constraints for arbitrary SU(N) groups then take the form

$$\hat{H} = \frac{g_L^2 d}{2a^2} \{ \hat{J}_R^a(A) \hat{J}_R^a(A) + 2 \hat{J}_R^a(A) \hat{J}_R^a(U) \} + \frac{g_L^2 L}{2a^2} \hat{J}_R^a(U) \hat{J}_R^a(U) \quad (21)$$

$$0 = \{ J_L^a(U) - J_R^a(U) - J_R^a(A) + \psi^\dagger(0) T^a \psi(0) \} | phys \rangle \quad (22)$$

$$0 = \{ J_L^a(A) + \psi^\dagger(m) T^a \psi(m) \} | phys \rangle \quad . \quad (23)$$

For comparison with the continuum formulation we next transform to the axial gauge. On the lattice this means we eliminate \hat{A} and we diagonalize \hat{U} , which can be achieved by applying two unitary transformations K_1 and K_2

$$K_1 = \exp(-i \rho_m^a \omega_A^a) , \quad \rho_m^a = \psi^\dagger(m) T^a \psi(m) \quad (24)$$

$$K_2 = \exp(-i q^a \Delta^a) , \quad q^a = \psi^\dagger(m) T^a \psi(m) + \psi^\dagger(0) T^a \psi(0) , \quad (25)$$

where the angles ω_A^a and Δ^a are related to A and U as

$$A = \exp(i \omega_A^a T^a), \quad U = \exp(i \Delta) \exp(i \theta) \exp(-i \Delta), \quad (26)$$

with $\exp(i\theta)$ diagonal. For later use we introduce auxiliary matrices R and P by

$$R_{ab} = 2Tr \left[\exp(-i\Delta) T^a \exp(i\Delta) T^b \right] \quad (27)$$

$$P_{ab} = 2Tr \left[\exp(-i\theta) T^a \exp(i\theta) T^b \right] - \delta_{ab} . \quad (28)$$

Transforming the constraints (22,23) we find²

$$J_R^a(A) | phys >_K = 0, \quad R_{ba1} J_R^b(U) | phys >_K = 0, \quad q^{a0} | phys >_K = 0, \quad (29)$$

where the subscript K on physical states signals that these states differ from the original ones by application of the unitary transformation. Using these constraints the unitarily transformed hamiltonian $\hat{H}' = K_2 K_1 \hat{H} K_1^\dagger K_2^\dagger$ reads in the physical Hilbert space

$$\begin{aligned} \hat{H}' = & \frac{g_L^2 d}{2a^2} \left\{ \rho_m^a \rho_m^a - 2\rho_m^{a0} \frac{i\partial}{\partial\theta^{a0}} + 2\rho_m^{a1} P_{a1b1}^{-1} q^{b1} \right\} + \frac{g_L^2 L}{2a^2} \left\{ P_{a1b1}^{-1} q^{b1} P_{a1c1}^{-1} q^{c1} \right. \\ & \left. - \frac{\partial^2}{\partial\theta^{a0} \partial\theta^{a0}} - \sum_{n < m=1}^N \cot \left(\frac{T_{nn}^{a0} - T_{mm}^{a0}}{2} \theta^{a0} \right) (T_{nn}^{a0} - T_{mm}^{a0}) \frac{\partial}{\partial\theta^{a0}} \right\} . \end{aligned} \quad (30)$$

The hamiltonian describes the dynamics of $N - 1$ physical variables θ^{a0} in the presence of the constraints on the fermionic charges eq.(29). According to these constraints we have to consider N basis states, one "singlet" state and $(N - 1)$ states in the adjoint representation.

In $SU(2)$ there is only one variable (θ^3) and the states can be classified as "singlet" $|S >$ and "triplet" $|T >$ from which general physical states are formed by linear superposition

$$|S > = \frac{1}{\sqrt{2}} \psi^\dagger(m) \psi(0) |0 >, \quad |T > = \frac{1}{\sqrt{2}} \psi^\dagger(m) 2T^3 \psi(0) |0 > , \quad (31)$$

$$|phys >_K = \mathcal{A}_S(\theta^3) |S > + \mathcal{A}_T(\theta^3) |T > . \quad (32)$$

The hamiltonian (30) written in two-component form then reads

$$H_{eff} = -\frac{g_L^2 L}{2a^2} \left[\frac{\partial^2}{\partial\theta^3} + \cot(\theta^3/2) \frac{\partial}{\partial\theta^3} \right] + \frac{g_L^2}{8a^2} \left\{ \begin{array}{cc} 3d & -4id \cot(\theta^3/2) - 4id\partial/\partial\theta^3 \\ -4id\partial/\partial\theta^3 & -d + 2L \sin^{-2}(\theta^3/2) \end{array} \right\} . \quad (33)$$

The unusual appearance of the differential operators in the off-diagonal elements can be corrected for by a third unitary transformation K_3 of the form

$$K_3 = \exp(i\rho_m^3 \theta^3 \frac{d}{L}) . \quad (34)$$

Applying this operator and rescaling $\theta^3 = 2\pi c$ we obtain the continuum hamiltonian as well as the basis states which were used in [3] to determine the spectrum of states

$$\begin{aligned} \tilde{H}_{eff} &= -\frac{g_L^2}{8a^2} \left[\frac{L}{\pi^2} \frac{\partial^2}{\partial c^2} + 2\pi \cot(\pi c) \frac{\partial}{\partial c} + \frac{d^2}{L} - \left\{ \begin{array}{cc} 3d & -2id \cot(\pi c) \\ 2id \cot(\pi c) & -d + 2L \sin^{-2}(\pi c) \end{array} \right\} \right] \\ |\tilde{S} > &= \frac{1}{\sqrt{2}} \psi^\dagger(m) \exp(2iT^3 \pi c \frac{d}{L}) \psi(0) |0 >, \quad |\tilde{T} > = \frac{1}{\sqrt{2}} \psi^\dagger(m) \exp(2iT^3 \pi c \frac{d}{L}) 2T^3 \psi(0) |0 > . \end{aligned}$$

Thus we have not only shown that identical results are found in continuum and lattice approach but also obtained the means to translate the results expressed in one language into the other. For a general $SU(N)$ group this allows us to identify the variables θ^{a0} (for $SU(2)$ the variable c) according to eq.(26) as a parametrization of the trace of the Wilson loop U winding around the circle in 1+1 dimensions. In higher dimensions these zero modes become "lower dimensional fields" maintaining the coordinate dependence in the orthogonal directions.

²Note that we use indices with subindex 0 to enumerate the elements belonging to the Cartan subalgebra and indices with subindex 1 to enumerate the remaining group elements.

5 Discussion and conclusion

For a discussion of the infrared properties of QCD_{1+1} we observe that the hamiltonians eqs.(30,33) have characteristic singularities in the variables θ^{a0} similar to centrifugal barriers. The "singlet" state eq.(31), which corresponds to the state $n=0$ in eq.(16), is the only eigenstate of the hamiltonian which has no θ^3 dependence and no L -dependent energy. Due to the singularities in the hamiltonian, the θ^3 dynamics is responsible for the L -dependent level splitting between the "singlet" and the remaining states. It is, however, not the origin of the d -dependent energy of the "singlet" state which can be traced back to the operator $\rho_m^a \rho_m^a$ in eq.(30) which would be found in QED, as well.

In 2+1 or 3+1 dimensions the axial gauge hamiltonian shows singularities in complete analogy to the 1+1 dimensional hamiltonian (30). The difference being the replacement of the zero modes θ^{a0} by "lower dimensional fields" θ^{a0} which depend on one or two spatial variables respectively. This analogy, however, is not sufficient for arguing that the "lower dimensional fields" will be as crucial as they are in 1+1 dimensions to understand the infrared properties and in particular confinement of QCD in 2+1 or 3+1 dimensions. The reason can be understood, if we remember that an excitation of the θ^3 variable is the same as the presence of the Wilson loop U , which winds around the circle, in the lattice wave function. The coupling to this particular Wilson loop is a necessity in 1+1 dimensions, if the fermions at sites $(0,d)$ do not couple to the gauge field string³ $P \exp[ig \int_0^d dx A(x)]$ in a way to form a singlet with respect to fermionic and gluonic color charge. In 2+1 or 3+1 dimensions our comparison with the lattice calculation explicitly reveals that it will be energetically favorable to couple to Wilson loops with an extension in directions orthogonal to the zero mode direction of the lower dimensional fields. In this way color singlet states may be formed without causing L -dependent energies. In QED these "transverse" degrees of freedom may be explicitly shown to change the linear potential to a Coulomb potential. Of course this does not exclude that the "lower dimensional fields" are important to understand the infrared properties of QCD. It tells, however, that the similarity in the singularity structure of the hamiltonian is not a guarantee for that.

Finally we want to mention a seemingly technical point. We observe that the calculation of the spectrum of states in the axial gauge hamiltonian eq.(30), in which all redundant variables are eliminated, is in general a very difficult task. In contrast, the lattice result eqs.(16,17) is easily generalized to any $\text{SU}(N)$ group. We believe that this is due to keeping unphysical degrees of freedom in the lattice formulation. It should therefore not be an artifact of the 1+1 dimensional model but rather point out the possible disadvantage of eliminating all unphysical degrees of freedom due to the complicated structure of the resulting Hamiltonian.

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³Here we make use of the continuum notation where P denotes path ordering

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